C4 JUNE 2013 NON-UC.

1. Express in partial fractions

$$\frac{5x+3}{(2x+1)(x+1)^2}$$

$$\frac{5x+3}{(2x+1)(x+1)^2} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\therefore 5x+3 = A(x+1)^2 + B(x+1)(2x+1) + C(2x+1)$$

$$x=-1 \Rightarrow -2 \equiv -C \quad \therefore \underline{C=2}$$

$$x=-\frac{1}{2} \Rightarrow \frac{1}{2} \equiv \frac{1}{4}A \quad \therefore \underline{A=2}$$

$$x=0 \Rightarrow 3 = A+B+C = 2+B+2 \quad \therefore \underline{B=-1}$$

$$\frac{2}{2x+1} - \frac{1}{x+1} + \frac{2}{(x+1)^2}$$

2. The curve C has equation

$$3^{x-1} + xy - y^2 + 5 = 0$$

Show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point (1, 3) on the curve C can be written in the form $\frac{1}{\lambda}\ln(\mu$

where λ and μ are integers to be found.

(7)

$$\frac{dy}{dx} \Rightarrow \quad 3^{x-1}\ln 3 + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$$

$$3^{x-1}\ln 3 + y = (2y - x)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{y + 3^{x-1}\ln 3}{2y - x}$$

$$y = 3 \Rightarrow \left.\frac{dy}{dx}\right|_{(1,3)} = \frac{3 + 3^0\ln 3}{2(3) - 1} = \frac{3 + \ln 3}{5}$$

$$\frac{dy}{dx} = \frac{1}{5}(\ln 3 + 3) = \frac{1}{5}(\ln 3 + \ln e^3)$$

$$= \frac{1}{5}\ln 3e^3 \qquad \lambda = 5, \mu = 3$$

3. Using the substitution $u = 2 + \sqrt{(2x + 1)}$, or other suitable substitutio… the value of

$$\int_0^4 \frac{1}{2 + \sqrt{(2x+1)}}\,dx$$

giving your answer in the form $A + 2\ln B$, where A is an integer and B is a positiv… constant.

(8)

$u = 2 + (2x+1)^{\frac{1}{2}}$

$x = 4 \quad u = 2 + 9^{\frac{1}{2}} = 5$

$x = 0 \quad u = 2 + 1^{\frac{1}{2}} = 3$

$\frac{du}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2 = \frac{1}{\sqrt{2x+1}}$

$\therefore \quad dx = \sqrt{2x+1}\; du.$

$\Rightarrow \int_3^5 \frac{1}{u}\sqrt{2x+1}\; du \Rightarrow \int_3^5 \frac{u-2}{u}\, du$

$= \int_3^5 1 - \frac{2}{u}\, du = \left[u - 2\ln u\right]_3^5$

$= (5 - 2\ln 5) - (3 - 2\ln 3)$

$= 2 + 2\ln 3 - 2\ln 5$

$= 2 + 2(\ln 3 - \ln 5)$

$= 2 + 2\ln\left(\frac{3}{5}\right) \qquad A = 2 \quad B = \frac{3}{5}$

4. (a) Find the binomial expansion of

$$\sqrt[3]{(8-9x)}, \qquad |x| < \frac{8}{9}$$

in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

(6)

(b) Use your expansion to estimate an approximate value for $\sqrt[3]{7100}$, giving your answer to 4 decimal places. State the value of x, which you use in your expansion, and show all your working.

(3)

a) $(8-9x)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(1-\frac{9}{8}x\right)^{\frac{1}{3}} = 2\left(1-\frac{9}{8}x\right)^{\frac{1}{3}}$

$= 2\left[1+\left(\frac{1}{3}\right)\left(-\frac{9}{8}x\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(-\frac{9}{8}x\right)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\left(-\frac{9}{8}x\right)^3\right]$

$= 2-\frac{3}{4}x-\frac{9}{32}x^2-\frac{45}{256}x^3$

b) $(8-9x)^{\frac{1}{3}}$ let $x=\frac{1}{10} \Rightarrow (8-9x)^{\frac{1}{3}} = 7.1^{\frac{1}{3}}$

$\therefore \left(\frac{1000(8-9x)}{1000}\right)^{\frac{1}{3}} \simeq \frac{\sqrt[3]{7100}}{10}$ when $x=0.1$

$\therefore \sqrt[3]{7100} \simeq 10\,(2-0.075-0.0028125-0.000175\ldots)$

$\simeq 10 \times 1.922011719$

$\simeq 19.22011719 \qquad = 19.2201$ (4dp)

5.

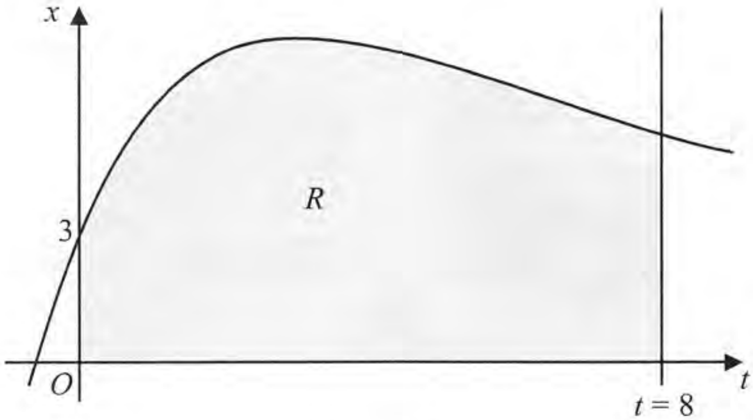


Figure 1

Figure 1 shows part of the curve with equation $x = 4te^{-\frac{1}{3}t} + 3$. The finite region R shown shaded in Figure 1 is bounded by the curve, the x-axis, the t-axis and the line $t = 8$.

(a) Complete the table with the value of x corresponding to $t = 6$, giving your answer to 3 decimal places.

t	0	2	4	6	8
x	3	7.107	7.218	6.248	5.223

(1)

(b) Use the trapezium rule with all the values of x in the completed table to obtain an estimate for the area of the region R, giving your answer to 2 decimal places.

(3)

(c) Use calculus to find the exact value for the area of R.

(6)

(d) Find the difference between the values obtained in part (b) and part (c), giving your answer to 2 decimal places.

(1)

a) $\frac{1}{2}(2)[3+5.223+2(7.107...)] \simeq 49.37$

b) $\int_0^8 4te^{-\frac{1}{3}t}+3\,dt$ $\quad u = 4t \quad v = -3e^{-\frac{1}{3}t}$

$u' = 4 \quad v' = e^{-\frac{1}{3}t}$

$\int uv' = uv - \int u'v$

$= [3t - 12te^{-\frac{1}{3}t}]_0^8 + \int_0^8 12e^{-\frac{1}{3}t}\,dt = [3t - 12te^{-\frac{1}{3}t} - 36e^{-\frac{1}{3}t}]_0^8$

$= [3t - 12te^{-\frac{1}{3}t}(t+3)]_0^8 = (24 - 132e^{-\frac{8}{3}}) - (-36) = 60 - 132e^{-\frac{8}{3}}$

d) $\therefore$ diff $= 1.0$

6. Relative to a fixed origin O, the point A has position vector $21\mathbf{i} - 17\mathbf{j} +$ [illegible] he p[illegible] has position vector $25\mathbf{i} - 14\mathbf{j} + 18\mathbf{k}$.

The line l has vector equation

$$\mathbf{r} = \begin{pmatrix} a \\ b \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ c \\ -1 \end{pmatrix}$$

where a, b and c are constants and λ is a parameter.

Given that the point A lies on the line l,

(a) find the value of a.

(3)

Given also that the vector $\overrightarrow{AB}$ is perpendicular to l,

(b) find the values of b and c,

(5)

(c) find the distance AB.

(2)

The image of the point B after reflection in the line l is the point B'.

(d) Find the position vector of the point B'.

(2)

$a = \begin{pmatrix} 21 \\ -17 \\ 6 \end{pmatrix}$ $b = \begin{pmatrix} 25 \\ -14 \\ 18 \end{pmatrix}$ a) $\begin{pmatrix} a+6\lambda \\ b+c\lambda \\ 10-\lambda \end{pmatrix} = \begin{pmatrix} 21 \\ -17 \\ 6 \end{pmatrix}$ $\therefore \lambda = 4$

$\underline{a = -3}$

b) $\overrightarrow{AB} = b - a = \begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ c \\ -1 \end{pmatrix} = 0$

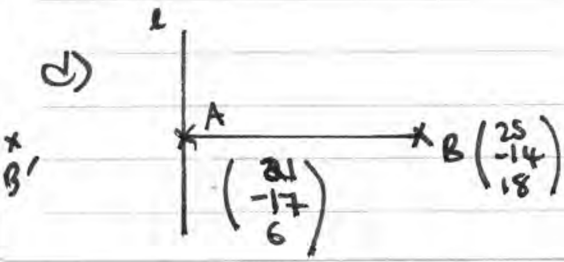
$24 + 3c - 12 = 0$

$\therefore \underline{c = -4}$

$b + (-4)(4) = -17$ $\therefore \underline{b = -1}$

c) $|\overrightarrow{AB}| = \sqrt{4^2 + 3^2 + 12^2} = \underline{13}$

d)



$\therefore B' = A - \overrightarrow{AB} = \begin{pmatrix} 21 \\ -17 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix}$

$B' \begin{pmatrix} 17 \\ -20 \\ -6 \end{pmatrix}$

7.

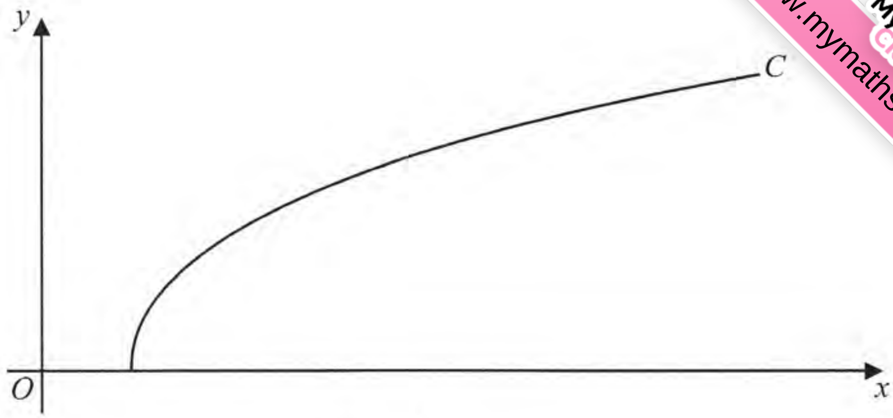


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 27\sec^3 t, \quad y = 3\tan t, \qquad 0 \leqslant t \leqslant \frac{\pi}{3}$$

(a) Find the gradient of the curve C at the point where $t = \dfrac{\pi}{6}$

(4)

(b) Show that the cartesian equation of C may be written in the form

$$y = (x^{\frac{2}{3}} - 9)^{\frac{1}{2}}, \qquad a \leqslant x \leqslant b$$

stating the values of a and b.

(3)

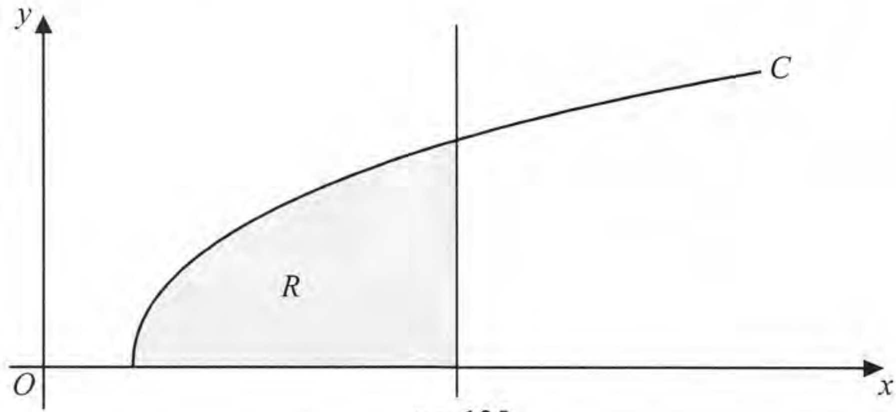


Figure 3

The finite region R which is bounded by the curve C, the x-axis and the line $x = 125$ is shown shaded in Figure 3. This region is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Use calculus to find the exact value of the volume of the solid of revolution.

(5)

a)

$x = 27(\sec t)^3$ $\qquad$ $y = 3t$

$\frac{dx}{dt} = 81(\sec t)^2 \times \sec t \tan t$ $\qquad$ $\frac{dy}{dt} = 3\sec$

$\frac{dx}{dt} = 81\sec^3 t \times \tan t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3\cancel{\sec^2 t}}{81\sec^{\cancel{3}} t \tan t} = \frac{1}{27\sec t \tan t}$

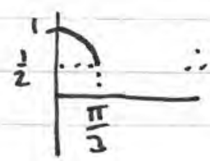
$\frac{dy}{dx} = \frac{\cos^2 t}{\sin t} = \frac{1-\sin^2 t}{\sin t}$ $\Rightarrow t = \frac{\pi}{6}$ $M_t = \frac{1-(\frac{1}{2})^2}{\frac{1}{2}} = \frac{3}{2}$

b) $y^2 = 9\tan^2 t$ $\qquad$ $\frac{x}{27} = (\sec t)^3 \Rightarrow \sec t = \frac{x^{\frac{1}{3}}}{3}$

$\therefore y^2 = 9(\sec^2 t - 1)$ $\qquad$ $\therefore \sec^2 t = \frac{x^{\frac{2}{3}}}{9}$

$\therefore y^2 = 9\left(\frac{x^{\frac{2}{3}}}{9} - 1\right) \Rightarrow y^2 = x^{\frac{2}{3}} - 9$ $\quad \therefore y = (x^{\frac{2}{3}} - 9)^{\frac{1}{2}}$

$x = \frac{27}{(\cos t)^3}$ $\qquad 0 \le t \le \frac{\pi}{3}$



$\therefore \frac{1}{2} \le \cos t \le 1$

$\frac{27}{(1)^3} \le x \le \frac{27}{(\frac{1}{2})^3}$ $\quad \therefore 27 \le x \le 216$

$a = 27 \quad b = 216$

c) $\text{Volume} = \pi\int_a^b y^2\,dx = \pi\int_{27}^{125} x^{\frac{2}{3}} - 9\,dx = \pi\left[\frac{3}{5}x^{\frac{5}{3}} - 9x\right]_{27}^{125}$

$= \pi\left[750 - -\frac{486}{5}\right] = \frac{4236\pi}{5}$

8. In an experiment testing solid rocket fuel, some fuel is burned and th... produ... collected. Throughout the experiment the sum of the masses of the u... waste products remains constant.

Let x be the mass of waste products, in kg, at time t minutes after the start of the ex... It is known that at time t minutes, the rate of increase of the mass of waste pro... in kg per minute, is k times the mass of unburned fuel remaining, where k is a posi... constant.

The differential equation connecting x and t may be written in the form

$$\frac{dx}{dt} = k(M - x), \text{ where } M \text{ is a constant.}$$

(a) Explain, in the context of the problem, what $\frac{dx}{dt}$ and M represent. **(2)**

Given that initially the mass of waste products is zero,

(b) solve the differential equation, expressing x in terms of k, M and t. **(6)**

Given also that $x = \frac{1}{2}M$ when $t = \ln 4$,

(c) find the value of x when $t = \ln 9$, expressing x in terms of M, in its simplest form. **(4)**

a) $\frac{dx}{dt}$ = rate of increase of the mass of waste

M = original Mass of unburned fuel.

b) $\int \frac{1}{M-x}\,dx = \int k\,dt \Rightarrow -\ln|M-x| = kt + c$

$t=0, x=0 \Rightarrow -\ln|M| = c \Rightarrow kt = \ln|M| - \ln|M-x|$

$\Rightarrow kt = \ln\left|\frac{M}{M-x}\right| \Rightarrow \frac{M}{M-x} = e^{kt} \Rightarrow M = (M-x)e^{kt}$

$\Rightarrow M = Me^{kt} - xe^{kt} \Rightarrow xe^{kt} = M(e^{kt}-1)$

$\therefore x = Me^{-kt}(e^{kt}-1) \Rightarrow x = M(1-e^{-kt})$

c) $\frac{1}{2}\cancel{M} = \cancel{M}\left(1 - e^{-k(\ln 4)}\right) \Rightarrow \left(e^{\ln 4}\right)^{-k} = \frac{1}{2}$

$\Rightarrow 4^{-k} = \frac{1}{2} \Rightarrow \ln 4^{-k} = \ln\left(\frac{1}{2}\right) \Rightarrow -k\ln 4 = -\ln$

$\Rightarrow k = \frac{\ln 2}{\ln 4}$

$x = M\left(1 - e^{-\frac{\ln 2}{\ln 4}\times \ln 9}\right)$

$\frac{\ln 2 \times \ln 3^2}{\ln 2^2} = \frac{\cancel{\ln 2} \times \cancel{2}\ln 3}{\cancel{2}\ln \cancel{2}}$

$= \ln 3$

$x = M\left(1 - e^{\ln 3^{-1}}\right) \Rightarrow x = M\left(1 - e^{\ln \frac{1}{3}}\right)$

$\Rightarrow x = M\left(1 - \frac{1}{3}\right) \therefore x = \frac{2}{3}M$